

APPENDIX

In this section we will derive the Potential function of our game-theoretic framework for distributed spectrum allocation in cognitive radio networks.

Let $\mathbb{N} = \{1, 2, \dots, N\}$ be set of Nodes and S_i be set of strategies available to node i and $i, j, k, l \in \mathbb{N}$ and $s_i \in S_i, s_j \in S_j$.

We define Interference function, $I_i(s_i, s_{-i})$ as:

$$I_i(s_i, s_{-i}) = \sum_{j=1, j \neq i}^N n_{i,j}^{-\gamma} f(s_i, s_j), \quad (\text{i})$$

where $s_{-i} = \{s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_N\}$, n_{ij} is number of hops between node i and j and γ is pathloss exponent,

$$\text{and } f(s_i, s_j) = \begin{cases} 1 & s_i = s_j, i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

We define Broadcast Latency function, $T_i(s_i, s_{-i})$ as:

$$T_i(s_i, s_{-i}) = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j), \quad (\text{ii})$$

where

$$\alpha_k = \begin{cases} 1 & \text{if } k \text{ is broadcast source} \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_{i,k} = \begin{cases} 1 & \text{if } i \text{ and } k \text{ are one-hop neighbours and } i \neq k \\ 0 & \text{otherwise,} \end{cases}$$

$$g(s_i, s_j) = 1 - f(s_i, s_j).$$

The utility function is defined as:

$$U_i(s_i, s_{-i}) = -w_{a,i} I_i(s_i, s_{-i}) - w_{b,i} T_i(s_i, s_{-i}). \quad (\text{iii})$$

Substitute $I_i(s_i, s_{-i})$ and $T_i(s_i, s_{-i})$ from (i) and (ii) in (iii),

$$U_i(s_i, s_{-i}) = -w_{a,i} \sum_{\substack{j=1 \\ j \neq i}}^N n_{i,j}^{-\gamma} f(s_i, s_j) - w_{b,i} \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j). \quad (\text{iv})$$

We define Potential Function as:

$$P(s_i, s_{-i}) = - \sum_{i=1}^N \left(w_{a,i} \sum_{\substack{j=1 \\ j \neq i}}^N \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) + w_{b,i} \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) \right).$$

$$P(s_i, s_{-i}) = -\sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) - \sum_{i=1}^N \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j). \quad (\text{v})$$

$$\text{Let } P(s_i, s_{-i}) = -M_1 - M_2, \quad (\text{vi})$$

where

$$M_1 = \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j), M_2 = \sum_{i=1}^N \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j).$$

We simplify M_1 as:

$$M_1 = \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j).$$

$$M_1 = \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) + \sum_{\substack{l=1 \\ l \neq i}}^N \sum_{\substack{j=1 \\ j \neq l}}^N w_{a,l} \lambda_a n_{l,j}^{-\gamma} f(s_l, s_j).$$

$$M_1 = \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) + \sum_{\substack{l=1 \\ l \neq i}}^N \left(\sum_{\substack{j=1 \\ j \neq l \\ j \neq i}}^N w_{a,l} \lambda_a n_{l,j}^{-\gamma} f(s_l, s_j) + w_{a,l} \lambda_a n_{l,i}^{-\gamma} f(s_l, s_i) \right).$$

$$M_1 = \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) + \sum_{\substack{l=1 \\ l \neq i}}^N \sum_{\substack{j=1 \\ j \neq l \\ j \neq i}}^N w_{a,l} \lambda_a n_{l,j}^{-\gamma} f(s_l, s_j) + \sum_{\substack{l=1 \\ l \neq i}}^N w_{a,l} \lambda_a n_{l,i}^{-\gamma} f(s_l, s_i).$$

$$\text{Let } Q(s_{-i}) = \sum_{\substack{l=1 \\ l \neq i}}^N \sum_{\substack{j=1 \\ j \neq l \\ j \neq i}}^N w_{a,l} \lambda_a n_{l,j}^{-\gamma} f(s_l, s_j),$$

$$\Rightarrow M_1 = \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) + \sum_{\substack{l=1 \\ l \neq i}}^N w_{a,l} \lambda_a n_{l,i}^{-\gamma} f(s_l, s_i) + Q(s_{-i}).$$

replacing l by j ,

$$M_1 = \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) + \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,j} \lambda_a n_{j,i}^{-\gamma} f(s_j, s_i) + Q(s_{-i}).$$

Since $f(s_j, s_i) = f(s_i, s_j)$, $n_{j,i} = n_{i,j}$,

$$\therefore M_1 = \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) + \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,j} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) + Q(s_{-i}).$$

$$M_1 = \sum_{\substack{j=1 \\ j \neq i}}^N \left(w_{a,i} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) + w_{a,j} \lambda_a n_{i,j}^{-\gamma} f(s_i, s_j) \right) + Q(s_{-i}).$$

$$M_1 = \sum_{\substack{j=1 \\ j \neq i}}^N \lambda_a (w_{a,i} + w_{a,j}) n_{i,j}^{-\gamma} f(s_i, s_j) + Q(s_{-i}).$$

$$\text{if } \lambda_a = \frac{w_{a,i}}{w_{a,i} + w_{a,j}}, \quad (\text{vii})$$

$$\Rightarrow M_1 = \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} n_{i,j}^{-\gamma} f(s_i, s_j) + Q(s_{-i}). \quad (\text{viii})$$

Now we simplify M_2 as:

$$M_2 = \sum_{i=1}^N \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j).$$

$$M_2 = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + \sum_{\substack{l=1 \\ l \neq i}}^N \sum_{\substack{k=1 \\ k \neq l}}^N \sum_{\substack{j=1 \\ j \neq l \\ j \neq k}}^N w_{b,l} \lambda_b \alpha_k \beta_{l,k} \beta_{k,j} g(s_l, s_j).$$

$$M_2 = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j)$$

$$+ \sum_{\substack{l=1 \\ l \neq i}}^N \sum_{\substack{k=1 \\ k \neq l}}^N \left(\sum_{\substack{j=1 \\ j \neq l \\ j \neq k \\ j \neq i}}^N \{ w_{b,l} \lambda_b \alpha_k \beta_{l,k} \beta_{k,j} g(s_l, s_j) \} + w_{b,l} \lambda_b \alpha_k \beta_{l,k} \beta_{k,i} g(s_l, s_i) \right).$$

$$M_2 = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j)$$

$$+ \sum_{\substack{l=1 \\ l \neq i}}^N \sum_{\substack{k=1 \\ k \neq l}}^N \sum_{\substack{j=1 \\ j \neq l \\ j \neq k \\ j \neq i}}^N w_{b,l} \lambda_b \alpha_k \beta_{l,k} \beta_{k,j} g(s_l, s_j) + \sum_{\substack{l=1 \\ l \neq i}}^N \sum_{\substack{k=1 \\ k \neq l}}^N w_{b,l} \lambda_b \alpha_k \beta_{l,k} \beta_{k,i} g(s_l, s_i).$$

$$\text{Let } O(s_{-i}) = \sum_{\substack{l=1 \\ l \neq i}}^N \sum_{\substack{k=1 \\ k \neq l}}^N \sum_{\substack{j=1 \\ j \neq l \\ j \neq k \\ j \neq i}}^N w_{b,l} \lambda_b \alpha_k \beta_{l,k} \beta_{k,j} g(s_l, s_j),$$

$$\Rightarrow M_2 = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + \sum_{\substack{l=1 \\ l \neq i}}^N \sum_{\substack{k=1 \\ k \neq l}}^N w_{b,l} \lambda_b \alpha_k \beta_{l,k} \beta_{k,i} g(s_l, s_i) + O(s_{-i}).$$

Replace l by j ,

$$M_2 = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{\substack{k=1 \\ k \neq j}}^N w_{b,j} \lambda_b \alpha_k \beta_{j,k} \beta_{k,i} g(s_j, s_i) + O(s_{-i}).$$

$\because g(s_i, s_j) = g(s_j, s_i), \beta_{i,k} = \beta_{k,i}$ and $\beta_{k,j} = \beta_{j,k},$

$$M_2 = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N \sum_{\substack{k=1 \\ k \neq j}}^N w_{b,j} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + O(s_{-i}).$$

Also $w_{b,j} \lambda_b \alpha_k \beta_{i,k} \beta_{k,i} g(s_j, s_i) = 0 \because \beta_{i,k} = 0$ for $i = k$, Therefore such term can be ignored,

$$M_2 = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + \sum_{\substack{j=1 \\ j \neq i \\ k \neq i}}^N \sum_{\substack{k=1 \\ k \neq j}}^N w_{b,j} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + O(s_{-i}).$$

Changing order of summation for second term:

$$M_2 = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,j} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + O(s_{-i}).$$

Also $w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) = 0 \because \beta_{k,j} = 0$ for $j = k$, Therefore such term can be ignored,

$$M_2 = \sum_{\substack{k=1 \\ k \neq j \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + \sum_{\substack{k=1 \\ k \neq j \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,j} \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + O(s_{-i}).$$

$$M_2 = \sum_{\substack{k=1 \\ k \neq i \\ k \neq j}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N (w_{b,i} + w_{b,j}) \lambda_b \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + O(s_{-i}).$$

$$\text{Let } \lambda_b = \frac{w_{b,i}}{w_{b,i} + w_{b,j}}, \quad (\text{ix})$$

$$\Rightarrow M_2 = \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) + O(s_{-i}). \quad (\text{x})$$

Put values of M_1 and M_2 from (viii) and (x) in (v),

$$P(s_i, s_{-i}) = - \sum_{\substack{j=1 \\ j \neq i}}^N w_{a,i} n_{i,j}^{-\gamma} f(s_i, s_j) - Q(s_{-i}) - \sum_{\substack{k=1, \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N w_{b,i} \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) - O(s_{-i}).$$

$$P(s_i, s_{-i}) = -w_{a,i} \sum_{\substack{j=1 \\ j \neq i}}^N n_{i,j}^{-\gamma} f(s_i, s_j) - w_{b,i} \sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) - O(s_{-i}) - Q(s_{-i}). \quad (\text{xi})$$

Using Equation (iv) and (xi):

$$P(s_i, s_{-i}) = U_i(s_i, s_{-i}) - O(s_{-i}) - Q(s_{-i}),$$

Hence $P(s_i, s_{-i})$ satisfy the follwing condition of exact potential function,

$$P(s'_i, s_{-i}) - P(s_i, s_{-i}) = U_i(s'_i, s_{-i}) - U_i(s_i, s_{-i}).$$

Therefore $P(s_i, s_{-i})$ is an exact potential function

Substituting, values of λ_a and λ_b from (vii) and (ix) in (v), potential is shown to be:

$$P(s_i, s_{-i}) = - \sum_{i=1}^N \left(\sum_{\substack{j=1 \\ j \neq i}}^N \frac{w_{a,i}^2}{w_{a,i} + w_{a,j}} n_{i,j}^{-\gamma} f(s_i, s_j) \right) - \sum_{i=1}^N \left(\sum_{\substack{k=1 \\ k \neq i}}^N \sum_{\substack{j=1 \\ j \neq i \\ j \neq k}}^N \frac{w_{b,i}^2}{w_{b,i} + w_{b,j}} \alpha_k \beta_{i,k} \beta_{k,j} g(s_i, s_j) \right).$$